1) Find the expectations of $S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in the superposition state $\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$.

**Answer** The expectation is the inner product of $\chi$ with $S_x \chi$. $<S_x> = <\chi | S_x | \chi>$. Write this out as a vector and matrix expression

$$<S_x> = <\chi | S_x | \chi> = \frac{1}{\sqrt{2}} (-i \begin{pmatrix} 1 \\ i \end{pmatrix}) \left[ \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \right]$$

Note the complex conjugation applied to the left-sided vector within the normal dot product relation.

First, do the $S_x$ operation...

$$\left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} i \\ 1 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 \\ i \end{pmatrix} \right)$$

$$<S_x> = \frac{1}{\sqrt{2}} (-i \begin{pmatrix} 1 \\ i \end{pmatrix}) \frac{\hbar}{2 \sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

...then the dot product for the result

$$(-i \begin{pmatrix} 1 \\ i \end{pmatrix}) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 0$$

$$<S_x> = \frac{\hbar}{4} \cdot 0 = 0$$

Do the $S_y$ similarly...

$$<S_y> = \frac{1}{\sqrt{2}} (-i \ begins{pmatrix} 1 \\ i \end{pmatrix}) \left[ \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} (-i \ begins{pmatrix} 1 \\ i \end{pmatrix}) \left[ \frac{\hbar}{2 \sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$<S_y> = \frac{\hbar}{4} (-2) = \frac{\hbar}{2}$$

2) Label each level of the angular momentum state diagrams shown with the corresponding state in full Dirac notation. Check the boxes that could apply to each diagram.

|3, 3⟩
|3, 2⟩
|3, 1⟩
|3, 0⟩
|3, -1⟩
|3, -2⟩
|3, -3⟩

**Answer** The $S^2$ or $L^2$ quantum number $\ell$ or $s$ is the top $S_z$ or $L_z$ value, $m$ with $-\ell \leq m \leq \ell$. Labeling states as $|\ell, m⟩$, the only way to have unit steps of the stated condition is with the above assignment scheme. "Spin" angular momentum can have either integer or half-integer positions -- including zero. Orbit must be on the integer positions, including zero. Both need to step by 1 (unit of $\hbar$) between possible projection outcomes.
3) Use the "factorization" $L^2 = L_+ L_- - \hbar L_z + L_z^2$ to evaluate the expectation $\langle \ell, m | L_+ L_- | \ell, m \rangle$.

**Answer** Solve for $L_+ L_-$ in the expression for $L^2$, then evaluate it in the expectation $\langle \ell, m | L^2 + \hbar L_z - L_z^2 | \ell, m \rangle$

$$\langle \ell, m | L_+ L_- | \ell, m \rangle = \hbar^2 \langle \ell, m | \ell + 1 + m - m^2 | \ell, m \rangle$$

$$\langle \ell, m | L_+ L_- | \ell, m \rangle = \hbar^2 (\ell (\ell + 1) - m(m - 1))$$

The square root of this was where the $C-$ in $L_- | \ell, m \rangle = C- | \ell, m - 1 \rangle$ came from.

4) Write down five representations for the state function of a spin 1/2 particle with $\hbar \over 2$ as its z-component of angular momentum.

**Answer** Anything we have used in class, used by Griffiths, or commonly used by physicists would be OK. Some examples...

$$| 1/2, -1/2 \rangle; \quad \psi_{\downarrow}; \quad \chi_{\downarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad |dn \rangle; \quad dn; \quad d; \quad |- \rangle; \quad \downarrow$$