QM1) a) The possible results of measuring any angular momentum component of a spin $\frac{1}{2}$ system are up and down, $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. Since there is no preferred direction in space, any orientation is no different than along the x, y, or z axes.

b) The operator associated with measurement along an arbitrary axis can be found by assuming that the vector $\mathbf{S}$ operator is always defined by its components, as it is classically. The x-, y-, and z-components are $S_x = S \cdot \mathbf{i}$, $S_y = S \cdot \mathbf{j}$, and $S_z = S \cdot \mathbf{k}$, so the arbitrary $u$ component is $S_u = S \cdot \mathbf{u}$ where $\mathbf{u}$ is the unit vector pointing in the direction of the spin measurement apparatus. In the case of an angle $\theta$ from the z-axis toward the +x-axis,

$$u_\theta = i \sin(\theta) + k \cos(\theta)$$

$$S_\theta = S \cdot (i \sin(\theta) + k \cos(\theta)) = \sin(\theta) \cdot S_x + \cos(\theta) \cdot S_z = \frac{\hbar}{2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

c) For a known state, the probability of obtaining each possible measurement result is the square of the coefficient needed to express the known state in terms of eigenstates of the operator associated with the measurement. Continuing in the basis of $S_z$ eigenstates, the known state in this case is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This state needs to expressed as a linear combination of the eigenstates of $S_\theta$, which must first be found...

Solve for the eigenstates of $S_\theta = \frac{\hbar}{2} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$

After the usual algebra -- solve the eigenvalue relation. $\begin{pmatrix} S_\theta - \lambda \mathbf{1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \lambda = m \hbar$ ...

$$m = \frac{1}{2} \quad \chi_{\text{up}} = \frac{\sin(\theta)}{\sqrt{2(1 - \cos(\theta))}} = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}$$

$$m = -\frac{1}{2} \quad \chi_{\text{dn}} = \frac{-\sin(\theta)}{\sqrt{2(1 + \cos(\theta))}} = \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}$$

Keep in mind that these states are not unique: either one could be multiplied by any phase factor.
Measurement of $S_z$ could give either the "up" or "dn" eigenvalues. Given that the electron state is in the "up" state of the z-projection, we need to expand this known state, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, as a superposition of the x-projection operator eigenstates $\chi_{up}$ and $\chi_{dn}$.

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_{up} \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) \\ \sin \left( \frac{\theta}{2} \right) \end{pmatrix} + C_{dn} \begin{pmatrix} -\sin \left( \frac{\theta}{2} \right) \\ \cos \left( \frac{\theta}{2} \right) \end{pmatrix}
\]

(expansion)

The probabilities of obtaining the eigenvalues are the squares of these expansion coefficients:

\[
p_{up\theta} = \left( |C_{up}| \right)^2 \quad p_{dn\theta} = \left( |C_{dn}| \right)^2
\]

The coefficients can be found by solving the two equations for the two unknowns in the (expansion) above. Alternatively, since $\chi_{up}$ and $\chi_{dn}$ are orthonormal eigenstates, you can simply take the inner product of both sides of the expression (expansion) with either $\chi_{up}$ or $\chi_{dn}$:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_{up} \chi_{up} + C_{dn} \chi_{dn}
\]

\[
< \chi_{up} | \begin{pmatrix} 1 \\ 0 \end{pmatrix} > = C_{up} < \chi_{up} | \chi_{up} > + C_{dn} < \chi_{up} | \chi_{dn} >
\]

\[
\cos \left( \frac{\theta}{2} \right) = C_{up}
\]

\[
< \chi_{dn} | \begin{pmatrix} 1 \\ 0 \end{pmatrix} > = C_{up} < \chi_{dn} | \chi_{up} > + C_{dn} < \chi_{dn} | \chi_{dn} >
\]

\[
-\sin \left( \frac{\theta}{2} \right) = C_{dn}
\]

So we have

\[
C_{up} = \cos \left( \frac{\theta}{2} \right) \quad C_{dn} = -\sin \left( \frac{\theta}{2} \right)
\]

giving the probabilities

\[
p_{up\theta} = \cos \left( \frac{\theta}{2} \right)^2 \quad p_{dn\theta} = \sin \left( \frac{\theta}{2} \right)^2
\]

Note that these probabilities match the $\theta=0$ case (measure $S_z$), the $\theta=\pi/2$ case (measure $S_x$), and the $\theta=\pi$ case (measure $-S_z$). The above result shows the probability for any $\theta$!
QM2) For a fixed spin $s$, the states and operators will be vectors and matrices in a $2s+1$ dimensional space. The first step is to make sure the basis is understood. Here is the $S_z$ basis used last week's problem...

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The spin operators will be identical matrices as the $L_x$, $L_y$, and $L_z$ derived last week...

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The operator $S_{111}$ is defined such that $S_{111} = S \cdot u_{111}$ where $u_{111} = \frac{1 + j + k}{\sqrt{3}}$ is the unit vector pointing toward the $x=y=z > 0$ direction.

Here is tri-diagonal direction spin component matrix, $S_{111} = \frac{S_x + S_y + S_z}{\sqrt{3}}$, so

$$S_{111} = \frac{\hbar}{\sqrt{3}} \begin{pmatrix} 1 & \frac{1-i}{\sqrt{2}} & 0 \\ \frac{1+i}{\sqrt{2}} & 0 & \frac{1-i}{\sqrt{2}} \\ 0 & \frac{1+i}{\sqrt{2}} & -1 \end{pmatrix}$$

By hand, the eigenvalues can be found with careful evaluation of determinant of $S_{111} - \lambda \cdot I$...

$$\begin{vmatrix} 1 - \lambda & \frac{1-i}{\sqrt{2}} & 0 \\ \frac{1+i}{\sqrt{2}} & -\lambda & \frac{1-i}{\sqrt{2}} \\ 0 & \frac{1+i}{\sqrt{2}} & -1 - \lambda \end{vmatrix} = 3 \cdot \lambda - \lambda^3$$

so the eigenvalues of $S_{111}$ are $\frac{\hbar}{\sqrt{3}}$ times this, or $-\hbar$, 0, or $+\hbar$, as expected. A numerical evaluation in Mathematica uses the "Eigenvalues[]" function. Mathcad -- what I use to write up these solutions -- uses one called "eigenvals()"...
Note the $\frac{1}{\sqrt{3}}$ included with the matrix here, and also that a strictly numerical evaluation, like that done by Mathcad, always has a numerical error which may or may not be rounded off in the displayed result.

QM3) If you are in a given quantum state $\psi$, and ask "What is the probability of obtaining a value $q$?" as the result of a measurement, you have to (1) identify the operator associated with the measurement, and (2) find out how much of $|q\rangle$ is required to express $\psi$ as a superposition of the orthonormal eigenstates $|q_j\rangle$ of the operator associated with the measurement, i.e.,

$$
\text{If } \psi = \sum_j c_j |q_j\rangle \text{ then the probabilities of obtaining any of the values } q_j \text{ will be } p_j = (|c_j|)^2.
$$

The coefficients can be found by taking the inner product of $|q_j\rangle$ with $\psi$, i.e., $p_j = |\langle q_j | \psi \rangle|^2$.

In this problem, we are given the state at $t=0$, the $\chi_{\psi}$ eigenstate of the $S_x$ operator:

$$
\chi(0) = \begin{pmatrix}
\cos \left( \frac{\theta}{2} \right) \\
\sin \left( \frac{\theta}{2} \right)
\end{pmatrix}
$$

With the Hamiltonian $H = -\gamma \cdot B_0 \cdot S_z$, the $z$-eigenstate energies are $\pm \left( -\gamma \cdot B_0 \cdot \frac{\hbar}{2} \right)$, and the time-dependent $z$-eigenstates become

$$
\begin{pmatrix}
1 \\
0
\end{pmatrix} \cdot e^{i \varphi} \quad \begin{pmatrix}
0 \\
1
\end{pmatrix} \cdot e^{-i \varphi}
$$

with \( \varphi = \frac{E}{\hbar} = \frac{\gamma \cdot B_0}{2} \).

The given state -- as a function of time after the measurement -- is then

$$
\chi(t) = \begin{pmatrix}
\cos \left( \frac{\theta}{2} \right) \cdot e^{i \varphi} \\
\sin \left( \frac{\theta}{2} \right) \cdot e^{-i \varphi}
\end{pmatrix}
$$

$$
\varphi = \frac{\gamma \cdot B_0 \cdot t}{2}
$$
The expectations can then be evaluated. These are:

\[ S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ < S_x > = < \chi(t) | S_x | \chi(t) > 
\]

\[ < S_x > = \left( \cos \left( \frac{\theta}{2} \right) e^{-i\varphi} \sin \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \left[ \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \left( \cos \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \]

Note the complex conjugation on the left.

\[ < S_x > = \frac{\hbar}{2} \left( \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) e^{-2i\varphi} + \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) e^{2i\varphi} \right) \]

\[ < S_x > = \frac{\hbar}{2} \cos(2\varphi) = \frac{\hbar}{2} \cos(\gamma B_0 t) \]

\[ S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ < S_y > = < \chi(t) | S_y | \chi(t) > 
\]

\[ < S_y > = \left( \cos \left( \frac{\theta}{2} \right) e^{-i\varphi} \sin \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \left[ \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \left( \cos \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \]

\[ < S_y > = \frac{\hbar}{2} \left( -i \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) e^{-2i\varphi} + i \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) e^{2i\varphi} \right) \]

\[ < S_y > = \frac{\hbar}{2} \sin(2\varphi) = \frac{\hbar}{2} \sin(\gamma B_0 t) \]

\[ S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ < S_z > = < \chi(t) | S_z | \chi(t) > 
\]

\[ < S_z > = \left( \cos \left( \frac{\theta}{2} \right) e^{-i\varphi} \sin \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \left[ \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \left( \cos \left( \frac{\theta}{2} \right) e^{i\varphi} \right) \]

\[ < S_z > = \frac{\hbar}{2} \left( \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \right) \]

\[ < S_z > = \frac{\hbar}{2} \cdot \cos(\theta) \]

\[ < S_z > = \frac{\hbar}{2} \cdot \cos(\theta B_0 t) \]
These expectations, $\langle S_x \rangle = \frac{\hbar}{2} \cos(\gamma \cdot B_0 \cdot t)$, $\langle S_y \rangle = \frac{\hbar}{2} \sin(\gamma \cdot B_0 \cdot t)$, and $\langle S_z \rangle = \frac{\hbar}{2} \cos(\theta)$ are exactly what one would obtain from a classical spinning top with a magnetic moment related by the gyromagnetic ratio $\gamma$ precessing due to the torque produced by the magnetic field. All vector components of the quantum electron in a magnetic field — on average — follow the classical result! **This is in spite of only being able to measure one projection component.** The averages require repeating the single component measurement many times on an identically prepared initial state.