Oscillatory Gap Damping

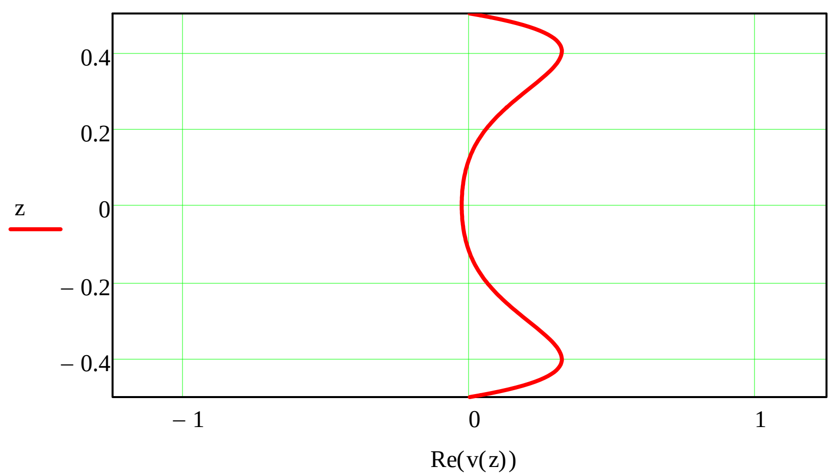
Find the damping due to the linear motion of a viscous gas in in a gap with an oscillating size: 1) Find the motion in a gap due to an oscillating external force; 2) Recast the solution in terms of an effective lateral bulk drag term; 3) Solve the Fourier-periodic solution driven by an equivalent uniform mass source.

1) Here's the solution for oscillatory flow in a gap driven by an externally applied force: $\frac{g}{2} < z < \frac{g}{2}$

$$\frac{d}{dt}v = D\cdot\frac{d^2}{dz^2}v + f_0\cdot e^{-i\cdot\omega\cdot t} \qquad D = \frac{\gamma}{\rho} \qquad f_0$$ external applied force per mass

$$v(z,t) = v_0\cdot\left(1 - \frac{\cosh(k\cdot z)}{\cosh\left(k\cdot\frac{g}{2}\right)}\right)\cdot e^{-i\cdot\omega\cdot t} \qquad k = \frac{1-i}{\delta} \qquad \delta = \sqrt{\frac{2\cdot\gamma}{\omega\cdot\rho}} \qquad v_0 = \frac{f_0}{-i\cdot\omega}$$

$$\delta := 0.125 \qquad \varphi := 2\cdot\pi\cdot FRAME \qquad z := -0.5, -0.495 \,..\, 0.5 \qquad v(z) := i\cdot\left[1 - \frac{\cosh\left[(1-i)\cdot\frac{z}{\delta}\right]}{\cosh\left(\frac{1-i}{2\cdot\delta}\right)}\right]\cdot\exp(-i\cdot\varphi)$$



For large drag (δ), the velocity is in phase with the force. Small drag has the velocity lagging the force by π/2 phase.

2) Now get the average velocity:

$$v_{av} = \frac{2}{g}\cdot\int_0^{\frac{g}{2}} v(z)\,dz = \frac{2}{g}\cdot\int_0^{\frac{g}{2}} v_0\cdot\left(1 - \frac{\cosh(k\cdot z)}{\cosh\left(k\cdot\frac{g}{2}\right)}\right)dz = v_0\cdot\left(1 - \frac{2}{k\cdot g}\cdot\tanh\left(k\cdot\frac{g}{2}\right)\right)$$

Describe by a bulk drag coefficient b (drag force per mass)

$$\frac{dv_{av}}{dt} = -b\cdot v_{av} + f_0 \qquad v_{av} = \frac{f_0}{b - i\cdot\omega}$$

Choose b to force the two solutions agree:

$$\frac{f_0}{b - i\cdot\omega} = \frac{f_0}{-i\cdot\omega}\cdot\left(1 - \frac{2}{k\cdot g}\cdot\tanh\left(k\cdot\frac{g}{2}\right)\right)$$

$$b(\omega) = i\cdot\omega\cdot\left(1 - \frac{1}{1 - \frac{2\cdot\Delta}{1-i}\cdot\tanh\left(\frac{1-i}{2\cdot\Delta}\right)}\right) \qquad \Delta = \frac{\delta}{g} = \sqrt{\frac{2\cdot\gamma}{\omega\cdot\rho\cdot g^2}}$$

$$b(\Delta) = \frac{2\cdot\gamma}{\rho\cdot g^2}\cdot G(\Delta) \qquad G(\Delta) := \frac{i}{\Delta^2}\cdot\left(\frac{1}{1 - \frac{1-i}{2\cdot\Delta}\cdot\coth\left(\frac{1-i}{2\cdot\Delta}\right)}\right)$$

small Δ... $\quad G(\Delta) = \frac{1-i}{\Delta} \qquad b = \sqrt{\frac{2\cdot\gamma\cdot\omega}{\rho\cdot g^2}}\cdot(1-i) = \frac{\delta}{g}\cdot\omega\cdot(1-i)$

The imaginary part reduces the average equivalent flow by removing δ from the undamped flow.

large Δ... $\quad G(\Delta) = 6 \qquad b = \frac{12\cdot\gamma}{\rho\cdot g^2} \qquad$ large drag

Wave Propagating into the Gap

Now consider a 1D compressible wave propagation into in a narrow gap... $\quad e^{i\cdot(k\cdot x - \omega\cdot t)}$

fluid acceleration $\quad \frac{\delta v}{\delta t} = -\frac{1}{\rho}\cdot\frac{\delta P}{\delta x} - b\cdot v$

conservation of mass $$\frac{1}{c^2}\cdot\frac{\delta P}{\delta t} = -\rho\cdot\frac{\delta v}{\delta x}$$

$$-i\cdot\omega\cdot v = -\frac{i\cdot k}{\rho}\cdot P - b\cdot v \qquad \frac{-i\cdot\omega}{c^2}\cdot P = -\rho\cdot i\cdot k\cdot v$$

$$\begin{bmatrix} i\cdot k & (b - i\cdot\omega)\cdot\rho \\ -i\cdot\omega & i\cdot k\cdot\rho\cdot c^2 \end{bmatrix}\cdot\begin{pmatrix} P \\ v \end{pmatrix} = 0$$

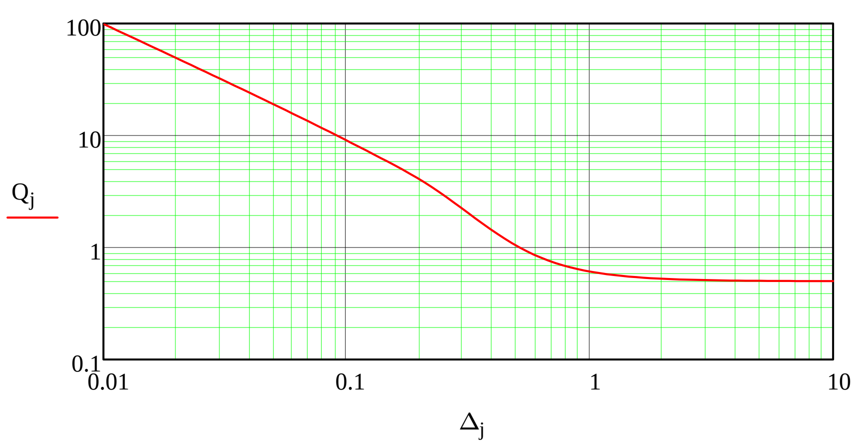
$$-k^2\cdot\rho\cdot c^2 + i\cdot\omega\cdot[(b - i\cdot\omega)\cdot\rho] = 0$$

$$k^2 = \frac{\omega^2 + i\cdot\omega\cdot b(\omega)}{c^2}$$

$$k = k_0\cdot\sqrt{1 + \frac{i\cdot b(\omega)}{\omega}} = k_0\cdot\sqrt{1 + i\cdot\Delta^2\cdot G(\Delta)} \qquad k_0 = \frac{\omega}{c}$$

$$Q(\Delta) = \frac{1}{2}\cdot\frac{Re(k)}{Im(k)}$$

$$j := 0..100 \qquad \Delta_j := .01\cdot\left(\frac{10}{.01}\right)^{\frac{j}{100}} \qquad rf_j := \sqrt{1 + i\cdot\left(\Delta_j\right)^2\cdot G\left(\Delta_j\right)} \qquad Q_j := \frac{1}{2}\cdot\frac{Re\left(rf_j\right)}{Im\left(rf_j\right)}$$



3) Finally, drive the motion by an equivalent mass influx uniformly distributed in the gap.

Rectangular Flow-Driven Gap

Gap oscillating as $g(t) = g\cdot\left(1 + \eta(x)\cdot e^{-i\cdot\omega\cdot t}\right)$ $\eta(x) = \eta_0$ in the gap

Convert the oscillating gap to a fixed gap with an equivalent oscillatory gas source.

fluid acceleration $$\frac{\delta v}{\delta t} = -\frac{1}{\rho}\cdot\frac{\delta P}{\delta x} - b(\omega)\cdot v$$

conservation of mass $$\frac{1}{c^2}\cdot\frac{\delta P}{\delta t} = -\rho\cdot\left(\frac{\delta v}{\delta x} + \frac{1}{g}\cdot\frac{\delta g}{\delta t}\right)$$

The driven solution with the source = 0 and the pressure =0 at the gap boundaries will be a Fourier series that is constant in the gap and opposite just outside the gap, periodically reversing for the Fourier series...

Square wave function, η_0 for $-\frac{L}{2} < x < \frac{L}{2}$

going to zero at the booundaties:

$$\eta(x) = 4\cdot\eta_0\cdot\sum_{(n=odd)}\left[\frac{(-1)^{\frac{n-1}{2}}}{n\cdot\pi}\cdot\cos\left(\frac{n\cdot\pi\cdot x}{L}\right)\right]$$

sum over n odd $P = P_n\cdot\cos\left(q_n\cdot x\right)\cdot e^{-i\cdot\omega\cdot t}$ $v = v_n\cdot\sin\left(q_n\cdot x\right)\cdot e^{-i\cdot\omega\cdot t}$ $\frac{\delta g}{\delta t} = -i\cdot\omega\cdot g\cdot\eta(x)\cdot e^{-i\cdot\omega\cdot t}$

$$-i\cdot\omega\cdot v_n\cdot\sin\left(q_n\cdot x\right) = \frac{1}{\rho}\cdot q\cdot P_n\cdot q_n\cdot\sin\left(q_n\cdot x\right) - b(\omega)\cdot v_n\cdot\sin\left(q_n\cdot x\right)$$

$$\frac{-i\cdot\omega}{c^2}\cdot P_n\cdot\cos\left(q_n\cdot x\right) = -\rho\cdot\left[q_n\cdot v_n\cdot\cos\left(q_n\cdot x\right) + \frac{1}{g}\cdot\left(-i\cdot\omega\cdot g\cdot\eta_n\cdot\cos\left(q_n\cdot x\right)\right)\right]$$

$$-i\cdot\omega\cdot v = \frac{1}{\rho}\cdot q\cdot P - b(\omega)\cdot v$$

Temporarily drop the n index.

$$\frac{-i\cdot\omega}{c^2}\cdot P = -\rho\cdot\left[q\cdot v + \frac{1}{g}\cdot(-i\cdot\omega\cdot g\cdot\eta)\right]$$

$$\begin{pmatrix} \frac{q}{\rho} & i\cdot\omega - b \\ -i\cdot\frac{\omega}{\rho\cdot c^2} & q \end{pmatrix}\cdot\begin{pmatrix} P \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ i\cdot\omega\cdot\eta \end{pmatrix}$$

$$\begin{pmatrix} P \\ v \end{pmatrix} = \frac{\begin{bmatrix} \left(\omega^2 + i\cdot\omega\cdot b\right)\cdot\rho\cdot c^2 \\ i\cdot\omega\cdot q\cdot c^2 \end{bmatrix}}{c^2\cdot q^2 - \omega^2 - i\cdot\omega\cdot b}\cdot\eta$$

Check out the case of b=0 for getting our bearings...

$$\begin{pmatrix} P \\ v \end{pmatrix} = \frac{\begin{pmatrix} \omega^2\cdot\rho\cdot c^2 \\ i\cdot\omega\cdot q\cdot c^2 \end{pmatrix}}{c^2\cdot q^2 - \omega^2}\cdot\eta$$

$$\eta(x) = 4\cdot\eta_0\cdot\sum_{(n=odd)}\left[\frac{(-1)^{\frac{n-1}{2}}}{n\cdot\pi}\cdot\cos\left(\frac{n\cdot\pi\cdot x}{L}\right)\right]$$

$$P(x) = 4\cdot\eta_0\cdot\omega^2\cdot\rho\cdot c^2\cdot\sum_{(n=odd)}\left[\frac{(-1)^{\frac{n-1}{2}}}{n\cdot\pi\cdot\left[c^2\cdot\left(\frac{n\cdot\pi}{L}\right)^2 - \omega^2\right]}\cdot\cos\left(\frac{n\cdot\pi\cdot x}{L}\right)\right]$$

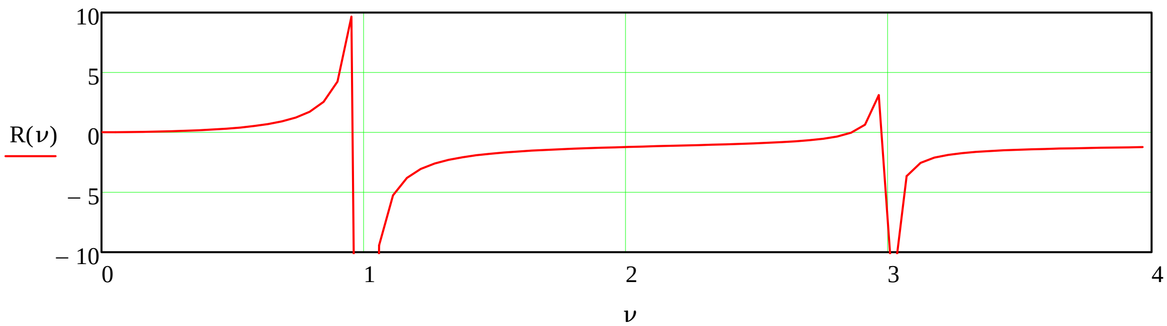
Force per length on the plate...

$$F = 2\cdot\int_0^{\frac{L}{2}} P(x)\,dx = 8\cdot L\cdot\eta_0\cdot\omega^2\cdot\rho\cdot c^2\cdot\sum_{(n=odd)}\left[\frac{1}{(n\cdot\pi)^2\cdot\left[c^2\cdot\left(\frac{n\cdot\pi}{L}\right)^2 - \omega^2\right]}\right]$$

$$F(\omega) = \frac{8\cdot L\cdot\eta_0\cdot\rho\cdot c^2}{\pi^2}\cdot\nu^2\cdot\sum_{(n=odd)}\left[\frac{1}{n^2\cdot\left(n^2 - \nu^2\right)}\right] \qquad \nu = \frac{\omega\cdot L}{\pi\cdot c} \qquad \omega = \frac{\pi\cdot c\cdot\nu}{L}$$

$n := 1, 3 .. 100 \qquad R(\nu) := \nu^2 \cdot \sum_n \left[\frac{1}{n^2 \cdot (n^2 - \nu^2)} \right]$

$\nu := 0, .052875 .. 4$



At low frequencies, the gas flows into the gap for positive η. As the first resonance is approached, the amplitude of the gas in-flow increases, resulting in the positive pressure-to-η phase relation. After the resonance, the phase is reversed. This negative phase relation is what would be expected for trapped gas indergoing rarefaction (negative pressure) with increasing v.

Now include the damping term b...

$$P_n = \frac{(\omega^2 + i \cdot \omega \cdot b) \cdot \rho \cdot c^2}{c^2 \cdot q^2 - \omega^2 - i \cdot \omega \cdot b} \cdot \eta_n$$

$$P(x) = 4 \cdot \eta_0 \cdot (\omega^2 + i \cdot \omega \cdot b)^2 \cdot \rho \cdot c^2 \cdot \sum_{(n=odd)} \left[\frac{(-1)^{\frac{n-1}{2}}}{n \cdot \pi \cdot \left[c^2 \cdot \left(\frac{n \cdot \pi}{L} \right)^2 - \omega^2 - i \cdot \omega \cdot b \right]} \cdot \cos\left(\frac{n \cdot \pi \cdot x}{L} \right) \right]$$

Force per length on the plate...

$$F = 2 \cdot \int_0^{\frac{L}{2}} P(x)\, dx = 8 \cdot L \cdot \eta_0 \cdot (\omega^2 + i \cdot \omega \cdot b)^2 \cdot \rho \cdot c^2 \cdot \sum_{(n=odd)} \left[\frac{1}{(n \cdot \pi)^2 \cdot \left[c^2 \cdot \left(\frac{n \cdot \pi}{L} \right)^2 - \omega^2 - i \cdot \omega \cdot b \right]} \right]$$

$$\frac{b}{\omega} = \frac{2\cdot\gamma\cdot L}{\pi\cdot c\cdot\nu\cdot g^2}\cdot G(\Delta) = \Delta^2\cdot G(\Delta) \qquad 1 + i\cdot\Delta^2\cdot G(\Delta) = \frac{1}{1 - \frac{2\cdot\Delta}{1-i}\cdot\tanh\left(\frac{1-i}{2\cdot\Delta}\right)}$$

$$\nu^2\cdot\left(1 + i\cdot\frac{b}{\omega}\right) = \frac{\nu^2}{1 - \frac{2}{1-i}\cdot\sqrt{\frac{\nu_0}{\nu}}\cdot\tanh\left(\frac{1-i}{2}\cdot\sqrt{\frac{\nu}{\nu_0}}\right)} \qquad \Delta = \sqrt{\frac{\nu_0}{\nu}} \qquad \nu_0 = \frac{2\cdot\gamma\cdot L}{\pi\cdot c\cdot\rho\cdot g^2}$$

$$F(\omega) = \frac{8\cdot L\cdot\eta_0\cdot\rho\cdot c^2}{\pi^2}\cdot\frac{\nu^2}{1 - \frac{2}{1-i}\cdot\sqrt{\frac{\nu_0}{\nu}}\cdot\tanh\left(\frac{1-i}{2}\cdot\sqrt{\frac{\nu}{\nu_0}}\right)}\cdot\sum_{(n=odd)}\left[\frac{1}{n^2\cdot\left(n^2 - \frac{\nu^2}{1 - \frac{2}{1-i}\cdot\sqrt{\frac{\nu_0}{\nu}}\cdot\tanh\left(\frac{1-i}{2}\cdot\sqrt{\frac{\nu}{\nu_0}}\right)}\right)}\right.$$

Now put in some real numbers... $\quad p = \frac{P}{1\cdot ATm} \qquad p := 0.25$

Fits to NIST Thermodata

$$c(p) := 179.342\cdot\left(1 - \frac{p}{40.3}\right) \qquad \gamma(p) := 5.37\cdot10^{-6}\cdot\left(1 + \frac{p}{84}\right) \qquad \rho(p) := 4.405\cdot p\cdot\left(1 + \frac{p}{21.5}\right)$$

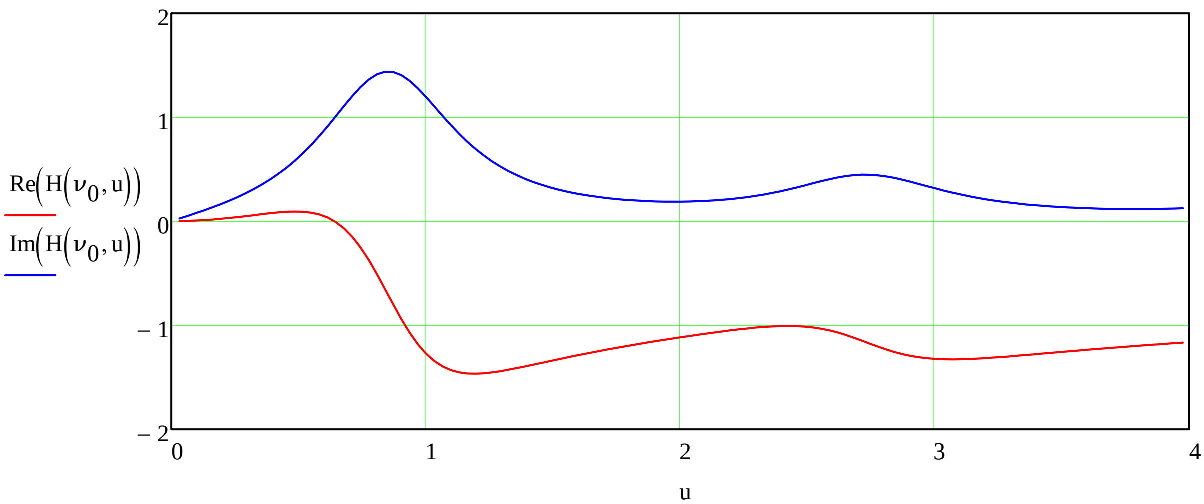
$$L := 0.003 \qquad w := 0.02 \qquad g := 20\cdot10^{-6} \qquad \omega := 2\cdot\pi\cdot34000 \qquad \nu := \frac{\omega\cdot L}{\pi\cdot c(p)} = 1.145$$

$$\nu_0 := \frac{2\cdot\gamma(p)\cdot L}{\pi\cdot c(p)\cdot\rho(p)\cdot g^2} \qquad \nu_0 = 0.13 \qquad M_p := \pi\cdot\left(0.0065^2 - 0.004^2\right)\cdot0.5\cdot3.$$

$$F = \frac{8\cdot L\cdot w\cdot\eta_0\cdot\rho\cdot c^2}{\pi^2}\cdot H(\nu)$$

$$H(\nu_0,\nu) := \left| \begin{array}{l} u \leftarrow \dfrac{\nu^2}{1 - \dfrac{2}{1-i}\cdot\sqrt{\dfrac{\nu_0}{\nu}}\cdot\tanh\left(\dfrac{1-i}{2}\cdot\sqrt{\dfrac{\nu}{\nu_0}}\right)} \\ \Sigma \leftarrow 0 \\ \text{for } n \in 1,3..83 \\ \quad \Sigma \leftarrow \Sigma + \dfrac{1}{n^2\cdot(n^2 - u)} \\ u\cdot\Sigma \end{array} \right.$$

$u := 0, .03231 .. 4$



This is the force with viscous loss, behaving the same as the b=0 case, but now with the imaginary part.

The real part of the scaled force is opposite to the displacement for our situation (around v=2. This is an extra restoring force due to the gas compression. The averagy loss comes from the positive imaginary part of the force:

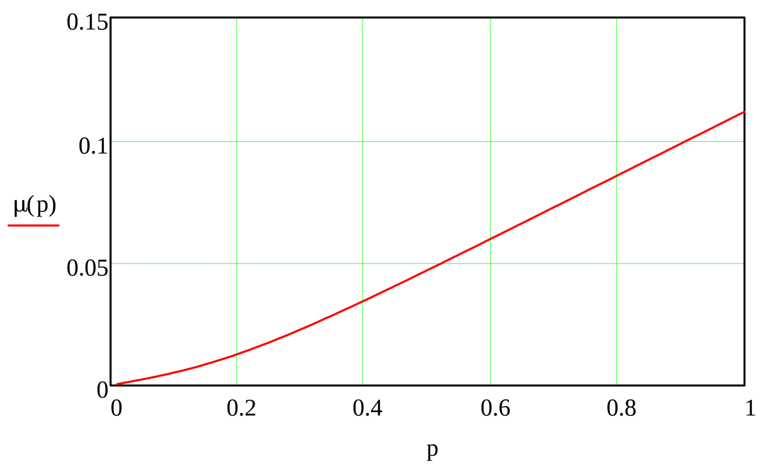
$p := 0.01, 0.02 .. 1$

Effective Mass $\qquad f = -M\cdot\omega^2\cdot x$

$$\frac{8\cdot L\cdot w\cdot\eta_0\cdot\rho\cdot c^2}{\pi^2}\cdot\mathrm{Re}\left(H\left(\nu_0,\nu\right)\right) = -M\cdot\omega^2\cdot g\cdot\eta$$

$$M = -\frac{8\cdot L^3\cdot w\cdot\rho}{\pi^4\cdot g}\cdot\frac{\mathrm{Re}\left(H\left(\nu_0,\nu\right)\right)}{\nu^2}$$

$$\mu(p) := -\frac{8\cdot L^3\cdot w\cdot\rho(p)}{\pi^4\cdot g\cdot M_p}\cdot\left|\begin{array}{l}\nu_0 \leftarrow \dfrac{2\cdot\gamma(p)\cdot L}{\pi\cdot c(p)\cdot\rho(p)\cdot g^2}\\ \nu \leftarrow \dfrac{\omega\cdot L}{\pi\cdot c(p)}\\ \dfrac{\mathrm{Re}\left(H\left(\nu_0,\nu\right)\right)}{\nu^2}\end{array}\right.$$



Damping

$$F = \frac{8\cdot L\cdot w\cdot\eta_0\cdot\rho\cdot c^2}{\pi^2}\cdot H(\nu)$$

$$W = -\frac{1}{2}\cdot\mathrm{Re}\left(F\cdot\overline{v}\right) = -\frac{1}{2}\cdot\mathrm{Re}\left[F\cdot\overline{\left(-i\cdot\omega\cdot\eta_0\cdot g\right)}\right] = \frac{1}{2}\cdot\omega\cdot\eta_0\cdot g\cdot\mathrm{Im}(F)$$

$$E = 2\cdot\frac{1}{2}\cdot M_p\cdot\left(\omega\cdot g\cdot\eta_0\right)^2$$

$$Q = \omega \cdot \frac{E}{W} = \omega \cdot \frac{2 \cdot \frac{1}{2} \cdot M_p \cdot (\omega \cdot g \cdot \eta_0)^2}{\frac{1}{2} \cdot \omega \cdot \eta_0 \cdot g \cdot \mathrm{Im}\left(\frac{8 \cdot L \cdot w \cdot \eta_0 \cdot \rho \cdot c^2}{\pi^2} \cdot H(\nu)\right)} = \frac{\pi^4 \cdot g \cdot M_p}{4 \cdot w \cdot \rho \cdot L^3} \cdot \frac{\nu^2}{\mathrm{Im}(H(\nu))}$$

$$Q(p) := \frac{\pi^4 \cdot g \cdot M_p}{4 \cdot w \cdot \rho(p) \cdot L^3} \cdot \left| \begin{array}{l} \nu_0 \leftarrow \dfrac{2 \cdot \gamma(p) \cdot L}{\pi \cdot c(p) \cdot \rho(p) \cdot g^2} \\ \nu \leftarrow \dfrac{\omega \cdot L}{\pi \cdot c(p)} \\ \dfrac{\nu^2}{\mathrm{Im}(H(\nu_0, \nu))} \end{array} \right.$$