Driven Disk Third Sound Resonator Properties
(MKS units unless otherwise noted)

Last Modified 02/09/2009

This is the outline of the transducer coupling through the disk-post resonator. See "Third Sound Resonator.xmcd" from the home page for the original gold cell" treatment from which this was derived.

The modes are assumed to be those of the free disk, but corrections are included for the non-zero disk width. See "disk post plane.xmcd" and "free disk.xmcd" for a comparison of the actual modes.

Contents

Bessel Functions

The primary basis mode is the rotating mode, used for easier manipulation of the transducer integrals.

$$\psi_r(r,\phi) = Jm(k_{mn} \cdot r) \cdot e^{i \cdot m \cdot \phi} \qquad k_{mn} = \frac{x_{mn}}{a}$$

The computational wave mode is the standing mode. All Bessel quantities will be expressed in terms of integrals over this function.

$$\psi_s(r,\phi) = Jm(k_{mn} \cdot r) \cdot \cos(m \cdot \phi)$$

The n^{th} zero of the derivitive of Jm(x) (s=1) or Jm(x) (s=-1) is needed everywhere...

$$x_{mn}(m,n,s) := \begin{array}{l} \text{if } s = 1 \\ \quad \left| \begin{array}{l} u \leftarrow \pi \cdot \left(n + .674 \cdot m^{.844} - .178 \cdot \dfrac{m^{.768}}{n} - .956 \right) \\ \text{root}\left(\dfrac{d}{du} Jn(m,u), u \right) \end{array} \right. \\ \text{otherwise} \\ \quad \left| \begin{array}{l} u \leftarrow \pi \cdot \left(n + .674 \cdot m^{.844} - .178 \cdot \dfrac{m^{.768}}{n} - .456 \right) \\ \text{root}(Jn(m,u), u) \end{array} \right. \end{array}$$

RMS value of $Jm(r)\cdot\cos(m\cdot\phi)$ over $0<r<x_{mn}$

$$J_{rms}(m,n,s) := \begin{cases} x \leftarrow x_{mn}(m,n,s) \\ \sqrt{\frac{1}{2}\cdot\left[1-\left(\frac{m}{x}\right)^2\right]\cdot Jn(m,x)^2} & \text{if } s = 1 \\ \frac{1}{\sqrt{2-\delta(m,0)}}\cdot\left|\frac{d}{dx}Jn(m,x)\right| & \text{otherwise} \end{cases}$$

Standing Wave...

film height

$$h(r,\phi,t) = h_0 + \eta\cdot Jn(m,k\cdot r)\cdot\cos(m\cdot\phi)\cdot\cos(\omega\cdot t)$$

flow field (peak flow is in the azimuthal component)

$$v(r,\phi,t) = c\cdot\frac{\eta}{h_0}\cdot\begin{bmatrix} -\frac{1}{k}\cdot\frac{d}{dr}(Jn(m,k\cdot r)\cdot\cos(m\cdot\phi)) \\ \frac{m}{k\cdot r}\cdot Jn(m,k\cdot r)\cdot\sin(m\cdot\phi) \end{bmatrix}\cdot\sin(\omega\cdot t)$$

particle displacement

$$\delta(r,\phi,t) = \frac{\eta}{k\cdot h_0}\cdot\begin{bmatrix} -\frac{1}{k}\cdot\frac{d}{dr}(Jn(m,k\cdot r)\cdot\cos(m\cdot\phi)) \\ \frac{m}{k\cdot r}\cdot Jn(m,k\cdot r)\cdot\sin(m\cdot\phi) \end{bmatrix}\cdot\cos(\omega\cdot t)$$

energy and angular momentum

$$E = \frac{1}{2}\cdot\left(\rho\cdot\pi\cdot a^2\cdot h_0\right)\cdot c^2\cdot\left(\frac{\eta}{h_0}\right)^2\cdot J_{rms}{}^2 \qquad L = 0$$

Travelling Wave...

film height

$$h(r,\phi,t) = h_0 + \eta\cdot Jn(m,k\cdot r)\cdot\cos(m\cdot\phi - \omega\cdot t)$$

flow field (peak flow is in the azimuthal component)

$$v(r,\phi,t) = c\cdot\frac{\eta}{h_0}\cdot\begin{bmatrix} \frac{1}{k}\cdot\frac{d}{dr}(Jn(m,k\cdot r)\cdot\sin(m\cdot\phi - \omega\cdot t)) \\ \frac{m}{k\cdot r}\cdot Jn(m,k\cdot r)\cdot\cos(m\cdot\phi - \omega\cdot t) \end{bmatrix}$$

particle displacement

$$\delta(r,\phi,t) = \frac{\eta}{k\cdot h_0}\cdot\begin{bmatrix} \frac{1}{k}\cdot\frac{d}{dr}(Jn(m,k\cdot r)\cdot\cos(m\cdot\phi - \omega\cdot t)) \\ -\frac{m}{k\cdot r}\cdot Jn(m,k\cdot r)\cdot\sin(m\cdot\phi - \omega\cdot t) \end{bmatrix}$$

energy and angular momentum

$$E = \left(\rho\cdot\pi\cdot a^2\cdot h_0\right)\cdot c^2\cdot\left(\frac{\eta}{h_0}\right)^2\cdot J_{rms}{}^2 \qquad L = \frac{m}{\omega}\cdot E$$

Bessel Functions

Physical Properties

$$\varepsilon_0 := 8.85\cdot 10^{-12} \qquad k := 1.3805\cdot 10^{-23}$$

Mass, dielectric constant, and density of liquid helium at low T...

$$m_4 := 6.646\cdot 10^{-27} \qquad \varepsilon_{He} := 1.055 \qquad \rho := 145.14$$

monolayer film thickness $h_1 := \left(\frac{m_4}{\rho}\right)^{\frac{1}{3}}$

Physical Properties

Third Sound Speed

See "third sound vs thickness.xmcd" for details.

substrate van der Waals constant $T_v := 38$

solid layer and retardation length (layers) $d_s := 1.46 \qquad \beta := 41.7$

$$c_3(h) := \left| \begin{array}{l} d \leftarrow \frac{h}{h_1} \\ c2 \leftarrow \left(1 - \frac{d_s}{d}\right)\cdot\frac{3\cdot k\cdot T_v}{m_4\cdot d^3} \\ \frac{\sqrt{c2\,\beta\cdot\left(\beta + \frac{4}{3}\cdot d\right)}}{\beta + d} \end{array} \right.$$

$$h(c) := \left| \begin{array}{l} x \leftarrow h_1\cdot\left(\frac{240000}{c^2}\right)^{\frac{1}{3}} \\ \mathrm{root}\left(c_3(x) - c, x\right) \end{array} \right.$$

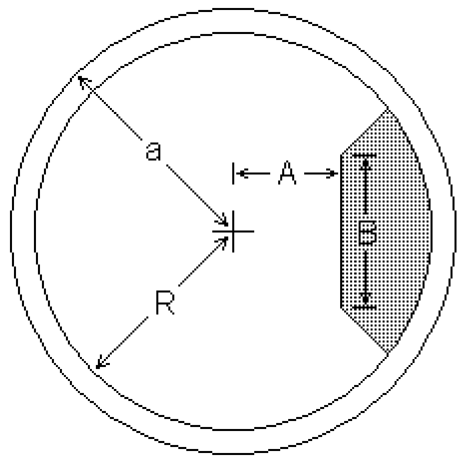
Third Sound Speed

Cell Properties

See "/stimulated condensation\pickup electrode shape.xmcd" for the original derivation.

The drive or pickup electrodes are assumed to be parallel plate gaps, neglecting any fringe fields. Each electrode is characterized by its area and a quantity (D or P below) reflecting its spatial overlap with the mode of interest. The active region of the electrodes are determined by the overlapping regions of the disk (at ground potential) and the electrode plates (applied or circuit potential).

The disk outer radius (a) and the capacitor outer radius (R) are different for two reasons: 1) The small sapphire disk thickness is most easily accounted for by a slightly inreased equivalent flet disk sise; and 2) The capacitor gap increases near the outer radius of the disk due to the deviation from flatness of the polished surface.



acutal disk oute radius $a_0 := 0.0065$

disk thickness $d := 0.0005$

effective flat disk radius $a := a_0 + \frac{1}{2} \cdot d$

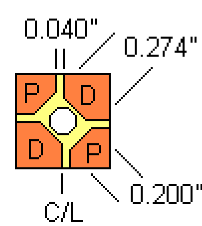
fraction of actual radius forming the small gap $x_{gap} := 0.92$ $a = 6.75 \times 10^{-3}$

effective capacitor outer radius $R := \frac{x_{gap} \cdot a_0}{a}$ $R = 0.886$

The "post cell" electrode geometry can be defined by two additional dimensions: The inner edge distance from the center (A) and the length of this inner edge (B). The outer edge is at the effective capacitor radius (R) and the sides extend at 45° between the inner edge and outer radius.

drive $A := \frac{0.200 \cdot 0.0254}{2 \cdot a}$ $B := 2 \cdot A$

pickup $A := \frac{0.274 \cdot 0.0254}{2 \cdot a}$ $B := 2 \cdot \left(A - \sqrt{2} \cdot \frac{0.040 \cdot 0.0254}{a} \right)$



$$A_p := \left| \begin{array}{l} C \leftarrow \dfrac{2 \cdot A - B + \sqrt{8 \cdot R^2 - (2 \cdot A - B)^2}}{4} \\ c \leftarrow \dfrac{C}{R} \\ 2 \cdot a^2 \cdot \left[(C - A) \cdot (B + C - A) + R^2 \cdot \left(\mathrm{acos}(c) - c \cdot \sqrt{1 - c^2} \right) \right] \end{array} \right.$$

C is the x value at the intersections of the 45º lines and the circle at R

$$A_p = 2.57 \times 10^{-5}$$

square meters, both drive plates in parallel

Drive and Pickup Integrals

These are the geometrical overlaps of the wave mode with the electrodes. For convenience, they are normalized to the full circle. Their use is derived in the "Mode Details" section..

$$D, P = \frac{1}{\pi \cdot a^2} \cdot \int J_m(k \cdot r) \cdot e^{i \cdot m \cdot \phi} \, dA$$

intergate over the regions exposed to the uniform gap of the parallel plate

pickup plate parameters

$$A := \frac{0.274 \cdot 0.0254}{2 \cdot a} \qquad B := 2 \cdot \left(A - \sqrt{2} \cdot \frac{0.040 \cdot 0.0254}{a} \right)$$

$$
P(m,n,s) := \left|\begin{array}{l}
xm \leftarrow x_{mn}(m,n,s) \\
rc \leftarrow \sqrt{A^2 + \left(\frac{B}{2}\right)^2} \\
\text{if } m > 0 \\
\quad \left|\begin{array}{l}
p1 \leftarrow \int_A^{rc} Jn(m, xm\cdot r)\cdot \sin\left(m\cdot \operatorname{acos}\left(\frac{A}{r}\right)\right)\cdot r\, dr \\
p2 \leftarrow \int_{rc}^{R} Jn(m, xm\cdot r)\cdot \sin\left[m\cdot\left(\frac{\pi}{4} - \operatorname{asin}\left(\frac{A - \frac{B}{2}}{\sqrt{2}\cdot r}\right)\right)\right]\cdot r\, dr \\
\text{if}\left[\operatorname{mod}(m,2) = 0, \frac{4}{m\cdot\pi}\cdot(p1 + p2), 0\right]
\end{array}\right. \\
\text{otherwise} \\
\quad \left|\begin{array}{l}
p1 \leftarrow \int_A^{rc} Jn(m, xm\cdot r)\cdot \operatorname{acos}\left(\frac{A}{r}\right)\cdot r\, dr \\
p2 \leftarrow \int_{rc}^{R} Jn(m, xm\cdot r)\cdot\left(\frac{\pi}{4} - \operatorname{asin}\left(\frac{A - \frac{B}{2}}{\sqrt{2}\cdot r}\right)\right)\cdot r\, dr \\
\frac{4}{\pi}\cdot(p1 + p2)
\end{array}\right.
\end{array}\right.
$$

pickup plates alligned along φ=0

both opposite plates -- extra factor of 2 if even, zero if odd.

m=0 equivalent to sin(mx)/m=x in the integrand

drive plate parameters $A := \frac{0.200\cdot 0.0254}{2\cdot a}$ $B := 2\cdot A$

drive plate alligned along the y axis (φ = π/2)

$$
D(m,n,s) := \left|
\begin{array}{l}
xm \leftarrow x_{mn}(m,n,s) \\
rc \leftarrow \sqrt{A^2 + \left(\frac{B}{2}\right)^2} \\
\text{if } m > 0 \\
\quad \left|
\begin{array}{l}
p1 \leftarrow \int_A^{rc} Jn(m, xm\cdot r)\cdot \sin\left(m\cdot \operatorname{acos}\left(\frac{A}{r}\right)\right)\cdot r\, dr \\
p2 \leftarrow \int_{rc}^{R} Jn(m, xm\cdot r)\cdot \sin\left[m\cdot\left(\frac{\pi}{4} - \operatorname{asin}\left(\frac{A - \frac{B}{2}}{\sqrt{2}\cdot r}\right)\right)\right]\cdot r\, dr \\
\frac{2\cdot \exp\left(i\cdot m\cdot \frac{\pi}{2}\right)}{m\cdot\pi}\cdot(p1 + p2)
\end{array}
\right. \\
\text{otherwise} \\
\quad \left|
\begin{array}{l}
p1 \leftarrow \int_A^{rc} Jn(m, xm\cdot r)\cdot \operatorname{acos}\left(\frac{A}{r}\right)\cdot r\, dr \\
p2 \leftarrow \int_{rc}^{R} Jn(m, xm\cdot r)\cdot\left(\frac{\pi}{4} - \operatorname{asin}\left(\frac{A - \frac{B}{2}}{\sqrt{2}\cdot r}\right)\right)\cdot r\, dr \\
\frac{2}{\pi}\cdot(p1 + p2)
\end{array}
\right.
\end{array}
\right.
$$

one plate only (D1) with the y-axis phase factor

m=0 equivalent to sin(mx)/m=x in the integrand

Gap Calibration

The transducer coupling requires knowledge of the electrode geometry. The area is assumed to be known, but the gap is deduced from the experimental capacitance. Experimentally, the LC oscillation frequency of the pickup detector (the TDO) is the accessible quantity. To get back to the pickup capacitance, we need to know the inductance (L) of the LC circuit and the amount of any stray capacitance NOT included as part of the gap. The inductance and physical areas are assumed to be the same as their room temperature determinations.

We thus have two unknow quantities that need to be experimentally determined: the gap and the stray capacitance. These unknowns are determined by the (1) the empty cell TDO frequency, and (2) the shift in the TDO frequency when only the gap is filled with liquid.

cell TDO inductance $L := 0.365\cdot 10^{-6}$

empty TDO frequency $f_{tdo} := 68390000$

shift when filled $\Delta f_{fill} := 1405000$ THIS NEEDS TO BE SET ACCORDING TO THE PARTICULAR EXPERIMENTAL RESULT

ratio of pickup capacitance to total $cpr := \frac{\left(\frac{f_{tdo}}{f_{tdo} - \Delta f_{fill}}\right)^2 - 1}{\varepsilon_{He} - 1}$ $cpr = 0.771$

capacitances $C_{tot} := \frac{1}{(2\cdot\pi\cdot f_{tdo})^2\cdot L}$ $C_p := cpr\cdot C_{tot}$ $C_s := C_{tot} - C_p$

pickup plates $C_p = 1.144 \times 10^{-11}$ other stray capacitances $C_s = 3.402 \times 10^{-12}$

capacitor gap $gap := \frac{\varepsilon_0\cdot A_p}{C_p}$ $gap = 1.989 \times 10^{-5}$

Cell Properties

Mode Details

The modes are specified by the angular node number m = 0, 1, 2..., the radial mode count n = 1, 2, 3... and a top-to-bottom symmetry parameter s = +1, or -1 for symmetric or antisymmetric respectively.

mode of interest $m := 2$ $n := 1$ $s := 1$ $xmn := x_{mn}(m, n, s)$

$jrms := J_{rms}(m, n, s)$

assumed drive voltage $V_d := 10$ V

In the following, the drive frequency f_{drive}, is the actual frequency of the drive oscillator. The applied force will be at twice this frequency due to the electrostatic frequency doubling..

fundamental resonance relations $\omega = c_3\cdot k$ $k = \frac{x_{mn}}{a}$ $c_3 = \frac{4\cdot\pi\cdot f_{drive}\cdot a}{xmn}$

Choose (a) or (b) below to specify the experimental situation...

(a) We are given the oscillator frequency on resonance...

$f_{drive} := 389$ Hz $\qquad c_3 := \dfrac{4\cdot\pi\cdot f_{drive}\cdot a}{xmn} \qquad c_3 = 10.803$ m/s

(b) We are given the third sound speed...

$c_3 := 11 \; \dfrac{m}{s}$ $\qquad$ frequency $\qquad f_{drive} := \dfrac{c_3\cdot xmn}{4\cdot\pi\cdot a} \qquad f_{drive} = 396.079$

assumed or measuresd quality factor $\qquad Q := 3000$

derived film thickness $\qquad h := h(c_3) \qquad h\cdot 10^9 = 4.036$ nm

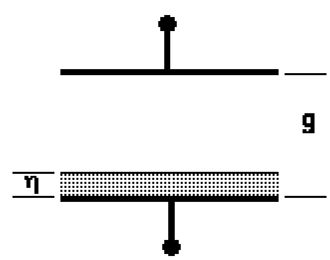
drive and pickup ovelaps (two drives) $\qquad D := 2\cdot D(m,n,s) \qquad P := P(m,n,s)$

$D = -0.071 \qquad P = 0.056$

The transducer coupling is found by calculating the DC response of the mode to a fixed voltage, then assuming a simple harmonic response to the AC drive.

The mode DC response to the drive is to be interpreted as the displacement of the mode, taken as a rigid motion, in response to the static electric field in the capacitor. The displacement is found by minimizing the total of the van der Waals energy and the electrostatic energy.

The capacitance change of a plate element with a film change η is found from the series combination of the vacuum and film section elements of the electrode. This is then integrated for the parallel capacitance of all the elements. This will be valid if the wavelength of the mode is much greater than the gap.



mode shape: $\qquad \eta(r,\phi) = \eta\cdot\psi(r,\phi)$

capacitance elements $\qquad dC = \dfrac{1}{\dfrac{1}{\dfrac{\varepsilon_0\cdot dA}{g-\eta}} + \dfrac{1}{\dfrac{\varepsilon\cdot dA}{\eta}}} - \dfrac{\varepsilon_0\cdot dA}{g} \qquad dC = \dfrac{\varepsilon_0}{g^2}\cdot\left(1-\varepsilon_{He}^{-1}\right)\cdot\eta\cdot dA$

The total change is the integral over drive or pickup, expressed in terms of D or P defined earlier

$$\Delta C = \frac{\varepsilon_0}{g^2}\cdot\left(1 - \varepsilon_{He}^{-1}\right)\cdot\eta\cdot\int \psi\, dA = \frac{\pi\cdot a^2\cdot\varepsilon_0}{g^2}\cdot\left(1 - \varepsilon_{He}^{-1}\right)\cdot\eta\cdot\begin{pmatrix} D \\ P \end{pmatrix}$$

The electrostatic energy change of the drive plate includes the work done by the voltage source at V_d.

$$\Delta U_{elec} = \frac{1}{2}\cdot\Delta C\cdot V_d^2 - V_d\cdot\Delta Q = -\frac{1}{2}\cdot\Delta C\cdot V_d^2 = -\frac{\pi\cdot a^2\cdot\varepsilon_0\cdot V_d^2}{2\cdot g^2}\cdot\left(1 - \varepsilon_{He}^{-1}\right)\cdot\eta\cdot D$$

The van der Waals energy is integrated over both the top and bottom of the disk. This is the "rigid mode assumption allowing the DC and AC responses to be connected by simple harmonic motion. .

$$\Delta U_{vdW} = \int \frac{1}{2}\rho\cdot f\cdot\eta^2\, dA = \rho\cdot f\cdot\eta^2\cdot\int \psi^2\, dA = \pi\cdot a^2\cdot\rho\cdot f\cdot\eta^2\cdot\psi_{rms}^2$$

over whole plate   over one face

$0 = \frac{d}{d\eta}\left(\Delta U_{elec} + \Delta U_{vdW}\right)$ determines the equilibrium mode displacement in the static field, η_{DC}

$$\eta_{DC} = \frac{1}{4}\cdot\frac{\varepsilon_0\cdot V_d^2\cdot\left(1 - \varepsilon_{He}^{-1}\right)\cdot D}{\rho\cdot f\cdot gap^2\cdot\psi_{rms}^2}$$

use $\quad c_3^2 = \frac{\rho_s}{\rho}\cdot f\cdot h \qquad \frac{\rho_s}{\rho} = 1 - \frac{d_s\cdot h_1}{h}$

$$\eta_{DC} := \frac{1}{4}\cdot\frac{\varepsilon_0\cdot V_d^2\cdot h\cdot\left(1 - \frac{d_s\cdot h_1}{h}\right)\cdot\left(1 - \varepsilon_{He}^{-1}\right)\cdot D}{\rho\cdot c_3^2\cdot gap^2\cdot jrms^2} \qquad \eta_{DC} = -6.085\times 10^{-15}\ \text{m}$$

The AC response has 1/2 the amplitude at twice the frequency of the drive, from $\cos(\theta)^2 = \frac{1+\cos(2\cdot\theta)}{2}$,

and the frequency dependence of a mass and spring:

$$\eta = \frac{\frac{1}{2}\cdot\eta_{DC}}{1-\left(\frac{\omega}{\omega_0}\right)^2-\frac{i}{Q}\cdot\frac{\omega}{\omega_0}}$$ $e^{-i\cdot\omega\cdot t}$ convention

On resonance, $\eta = \frac{i}{2}\cdot\eta_{DC}\cdot Q$

thickness oscillations on resonance

$$\eta_{mn} := \frac{i}{8}\cdot h\cdot Q\cdot\left(1-\frac{d_s\cdot h_1}{h}\right)\cdot\frac{\varepsilon_0\cdot V_d^{\,2}}{gap^2\cdot\rho\cdot c_3^{\,2}}\cdot\frac{\left(1-\varepsilon_{He}^{\,-1}\right)\cdot D}{jrms^2}$$

fractional oscillations $\frac{\eta_{mn}}{h} = -2.261i\times10^{-3}$

TDO frequency modulation comes from $\frac{\Delta f}{f} = -\frac{1}{2}\cdot\frac{\Delta C_p}{C}$ with $\Delta C_p = \frac{\pi\cdot a^2\cdot\varepsilon_0}{g^2}\cdot\left(1-\varepsilon_{He}^{\,-1}\right)\cdot\eta\cdot P$

$$\frac{\Delta f}{\eta} = -\frac{1}{2}\cdot\frac{f_{tdo}}{C_{tot}}\cdot\frac{\Delta C_p}{\eta}$$

This is the mode sensitivity of the LC oscillator $dfd\eta = -\frac{1}{2}\cdot\frac{f_{tdo}}{gap}\cdot\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\cdot\frac{C_p}{C_{tot}}\cdot\frac{\pi\cdot a^2}{A_p}\cdot P$

mode sensitivity $dfd\eta := -\frac{1}{2}\cdot\frac{f_{tdo}\cdot\left(1-\varepsilon_{He}^{\,-1}\right)\cdot cpr\cdot\pi\cdot a^2\cdot P}{gap\cdot A_p}$ $dfd\eta\cdot10^{-9} = -21.69$ $\frac{Hz}{nm}$

Here's the theoretical requency modulation of the TDO for all of the conditions specified previously...

$f_{mn} := \eta_{mn}\cdot dfd\eta$ $f_{mn} = 0.198i$

Here's the complete expression in one formula...

$$f_{mn} := -\frac{i}{16}\cdot Q\cdot f_{tdo}\cdot\frac{h}{gap}\cdot\left(1-\varepsilon_{He}^{\,-1}\right)^2\cdot cpr\cdot\frac{\pi\cdot a^2}{A_p}\cdot\frac{D\cdot P}{jrms^2}\cdot\left(1-\frac{d_s\cdot h_1}{h}\right)\cdot\frac{\varepsilon_0\cdot V_d^{\,2}}{gap^2\cdot\rho\cdot c_3^{\,2}}$$

$f_{mn} = 0.198i$

Finally, the actual comparison to experimentally measured voltage amplitudes requires a calibration of the complete FM detection system. This is the "PLLCAL", the response of the voltage measuring device to a known frequency modulation imposed in the detector electronics.

Here's the expected voltage amplitude with $\mathrm{PLLCAL} := 350 \quad \frac{\mu V}{Hz}$

$$A_{res} := \mathrm{PLLCAL} \cdot f_{mn} \qquad A_{res} = 69.296i \quad \mu V$$

PLLCAL is the calibration signal and A_{res} is the expected signal size on resonance..

Mode Details

Revision History

02/09/2009 - The P and D functions were modified to work for m=0 modes

Revision History