CAPACITIVE THIRD SOUND RESONATOR

Imagine that the mode in question is rigid in the sense that even at DC, the mode retains it's shape. Let the mode be described by an amplitude and a unitless mode function:

$$h(r,\phi,t) = h_0 + \eta \cdot \psi(r,\phi) \cdot e^{-i \cdot \omega \cdot t} \qquad (1)$$

If this is the case, the electrostatic energy change due to the mode within the drive plate is given by

$$\Delta U_e = \int_{drive} u_v \, dV = \int_{drive} \frac{1}{2} \cdot (\varepsilon - \varepsilon_0) \cdot E^2 \cdot \eta \cdot \psi(r,\phi) \, dA \qquad (2)$$

The Van der Waals energy change associated with the same mode is

$$\Delta U_v = \int_{cell} u_a \, dA = \int_{cell} \frac{1}{2} \cdot \rho \cdot f \cdot \eta^2 \cdot \psi^2 \, dA \qquad (3)$$

Equating the derivitives with respect to amplitude gives the DC mode amplitude, written in terms of the flat film DC thickness shift

$$\eta_{dc} = \eta_0 \cdot \frac{\int_{drive} \psi(r,\phi) \, dA}{\int_{cell} \psi^2 \, dA} \qquad \eta_0 = \frac{1}{2} \cdot \frac{(\varepsilon - \varepsilon_0) \cdot E^2}{\rho \cdot f} \qquad (4a,4b)$$

Now, turn on the AC excitation of the mode, noting that the AC oscillations are between 0 and η_{dc} , resulting in the factor of 1/2.

$$\eta = \frac{\frac{1}{2} \cdot \eta_{dc}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 - \frac{i \cdot \omega}{Q \cdot \omega_0}} \qquad (5)$$

With a pickup capacitor that responds to a uniform DC shift by df/dh, the mode will contribute according to its average height:

$$\Delta f = \frac{df}{dh} \cdot \eta \cdot \frac{1}{A_{pick}} \cdot \int_{pickup} \psi(r,\phi) \, dA \qquad (6)$$

The drive and pickup integrals are more conveniently expressed in terms of normalizations to the whole cell:

$$D = \frac{1}{A_{cell}} \cdot \int_{drive} \psi(r,\phi) \, dA \qquad P = \frac{1}{A_{cell}} \cdot \int_{pickup} \psi(r,\phi) \, dA \qquad (7)$$

Put everything together

$$\Delta f = \frac{df}{dh} \cdot P \cdot \frac{A_{cell}^{2}}{A_{pick} \cdot \int_{cell} \psi^2 \, dA} \cdot D \cdot \frac{\frac{1}{2} \cdot \eta_0}{1 - \left(\frac{\omega}{\omega_0}\right)^2 - \frac{i \cdot \omega}{Q \cdot \omega_0}} \tag{8}$$

If the cell is a circle, radius a, with plates symmetric about y=0, only the cosine modes couple...

$$\psi = J_m\left(k_{mn} \cdot r\right) \cdot \cos(m \cdot \phi) \tag{9a}$$

$$\frac{1}{A_{cell}} \cdot \int_{cell} \psi^2 \, dA = \frac{1}{\pi \cdot a^2} \cdot \int_0^a \int_0^{2 \cdot \pi} \left(J_m\left(k_{mn} \cdot r\right) \cdot \cos(m \cdot \phi)\right)^2 \cdot r \, dr \, d\phi$$

$$\frac{1}{A_{cell}} \cdot \int_{cell} \psi^2 \, dA = \frac{1}{a^2} \cdot \int_0^a J_m\left(k_{mn} \cdot r\right)^2 \cdot r \, dr = \frac{1}{2} \cdot \left[1 - \frac{m^2}{\left(x_{mn}\right)^2}\right] \cdot J_m\left(x_{mn}\right)^2 \tag{9b}$$

Including this

$$\Delta f = \frac{df}{dh} \cdot P \cdot \frac{A_{cell}}{A_{pick} \cdot \left[\left[1 - \frac{m^2}{\left(x_{mn}\right)^2}\right] \cdot J_m\left(x_{mn}\right)^2\right]} \cdot D \cdot \frac{\eta_0}{1 - \left(\frac{\omega}{\omega_0}\right)^2 - \frac{i \cdot \omega}{Q \cdot \omega_0}} \tag{10}$$

The pickup capacitor response is found from $\frac{\Delta C_p}{C_p} = 2 \cdot \frac{\Delta h}{g} \cdot \left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$ and $\frac{\Delta f}{f} = -\frac{1}{2} \cdot \frac{\Delta C_p}{C_{total}}$ giving

$$\frac{df}{dh} = -\frac{f_0}{g} \cdot \left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \cdot \frac{C_p}{C_{total}} \tag{11}$$

The "mode sensitivity" can also be written as

$$f_{mn} = \frac{df}{dh} \cdot \frac{1}{A_{pick}} \cdot \int_{pickup} \psi(r, \phi) \, dA = \frac{df}{dh} \cdot \frac{A_{cell}}{A_{pick}} \cdot P \tag{12}$$

Summarizing the results of Hai's thesis, for an arbitrary linear combination of the m modes, the D and P should be replaced by the two component vector dot products

$$\bar{a} \cdot D \qquad a \cdot P \tag{13}$$

The vector a is normalized and has components coresponding to the right (+m) and left (-m) polarized wave components so that the appropriate wave function is

$$\psi(r, \phi) = J_m\left(k_{mn} \cdot r\right) \cdot \left(a_m \cdot e^{i \cdot m \cdot \phi} + a_{-m} \cdot e^{-i \cdot m \cdot \phi}\right) \tag{14}$$

The integrals with ψ^2 in (4) and (8) remain integrals over the standing wave function (9a), not (14) and D and P have components given by,

$$D_m = \frac{1}{\sqrt{2}} \cdot \frac{1}{A_{cell}} \cdot \int_{drive} J_m\left(k_{mn} \cdot r\right) \cdot e^{-i \cdot m \cdot \phi} \, dA \qquad P_m = \frac{1}{\sqrt{2}} \cdot \frac{1}{A_{cell}} \cdot \int_{pick} J_m\left(k_{mn} \cdot r\right) \cdot e^{i \cdot m \cdot \phi} \, dA \qquad (15)$$

A few of the definitions are different from the notation in Hai's thesis:

1) D and P in (7) have been defined to be the same as the drive and pickup averages in the program fmncalc, and in (15) also include a "normalizing root" 2. The definitions in Hai's thesis also nomalize the integrals as in (4) and (6) as opposed to the cell area.

2) The DC film response (4b) to a field is incorrectly identified without the factor of 1/2 in Hai's thesis. The algebra in the thesis is still OK as he states it.

3) The drive integral in Hai's thesis includes a strength function G. Here it is assumed that the integral over the drive can be broken up to include drive plates with different phases.